

On Sandon-type metrics for contactomorphism groups

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Abstract

For certain contact manifolds admitting a 1-periodic Reeb flow we construct a conjugation-invariant norm on the (universal cover of) the contactomorphism group. With respect to this norm the group admits a quasi-isometric monomorphism of the real line. The construction involves the partial order on contactomorphisms and symplectic intersections. As a counterpoint, we show that no unbounded conjugation-invariant norms exist for the standard contact sphere.

1 Introduction

A *conjugation-invariant norm* on a group G is a function $\nu: G \rightarrow [0, \infty)$ satisfying the following properties:

1. $\nu(\mathbb{1}) = 0$ and $\nu(g) > 0$ for all $g \neq \mathbb{1}$.
2. $\nu(gh) \leq \nu(g) + \nu(h)$ for all $g, h \in G$.
3. $\nu(g^{-1}) = \nu(g)$ for all $g \in G$.
4. $\nu(h^{-1}gh) = \nu(g)$ for all $g, h \in G$.

Given any bi-invariant metric d on G , distance to the identity defines a conjugation-invariant norm, $\nu(f) := d(f, \mathbb{1})$, and vice versa, any

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conjugation-invariant norm ν defines a bi-invariant metric $d(f, g) := \nu(fg^{-1})$.

Following the terminology of [BIP08] we say a norm on G is *bounded* when there exists $C < \infty$ such that $\nu(g) \leq C$ for all $g \in G$. A norm is called *stably unbounded* if for some $g \in G$, $\nu(g^n) \geq c|n|$ for all $n \in \mathbb{Z}$ with some $c > 0$. A norm ν is *discrete* if there exists a constant $c > 0$ such that $c \leq \nu(g)$ for any $g \neq 1$.

In this paper, we focus on conjugation-invariant norms on contactomorphism groups and in particular on their (un)boundedness and discreteness. Such norms were discovered by S. Sandon in [S10] and further studied in recent papers [Z12] by F. Zapolsky and [CS12] by V. Colin and S. Sandon. Their geometric properties turn out to be sensitive to the contact topology of (V, ξ) . The above norms are:

- bounded for $T^*\mathbb{R}^n \times S^1$, but not stably unbounded [S10];
- stably unbounded for $T^*X \times S^1$ with compact X [Z12, CS12] and for $\mathbb{R}P^{2n+1}$; [CS12];
- bounded for S^{2n+1} [CS12],

where the manifolds in the list are equipped with the standard contact structures. All these norms are studied by using Legendrian spectral invariants. Sandon's norm and its extension by Zapolsky are actually defined through these invariants, while the Colin-Sandon norms admit a geometric and/or dynamical definitions. Let us mention also a work in progress [AM12] by P. Albers and W. Merry where the authors define a conjugation-invariant norm on contact manifolds admitting a periodic Reeb flow by means of Rabinowitz Floer homology.

The aim of the present paper is twofold. In Section 2 for certain contact manifolds admitting a 1-periodic Reeb flow we give yet another construction of a stably unbounded conjugation-invariant norm on the (universal cover of) contactomorphism groups. The construction involves the partial order on contactomorphism groups introduced in [EP00], and the stable unboundedness of the norm is deduced from basic results on symplectic intersections. The examples include various prequantization spaces such as $T^*X \times S^1$ with closed X , prequantizations of symplectically aspherical manifolds containing a closed Bohr-Sommerfeld Lagrangian submanifold and standard projective spaces $\mathbb{R}P^{2n+1}$.

These results are contrasted with the following statement proved in Section 3: every conjugation invariant norm on the identity component

of the contactomorphism group of the standard contact sphere S^{2n+1} is bounded and discrete. The proof follows closely [BIP08].

2 Constructions

Let (V, ξ) be a contact manifold, not necessarily closed, with co-oriented contact structure ξ . Let us fix some notation for the groups we will be dealing with: We write $\mathcal{G}_c(V, \xi)$ for the identity component of the group of compactly supported contactomorphisms of (V, ξ) . This is shortened to $\mathcal{G}_c(V)$ or \mathcal{G}_c when clear from the context. When V is closed we omit the subscript c and write simply \mathcal{G} . We denote by $\tilde{\mathcal{G}}_c$ the universal cover of \mathcal{G}_c .

Contact isotopies supported in a given open set $X \subset V$ give rise to subgroups of $\mathcal{G}_c(V)$ and $\tilde{\mathcal{G}}_c(V)$ generated by these isotopies. We denote them respectively by $\mathcal{G}_c(X) \subset \mathcal{G}_c(V)$ and $\tilde{\mathcal{G}}_c(X, V) \subset \tilde{\mathcal{G}}_c(V)$. Let us mention that in general $\tilde{\mathcal{G}}_c(X, V)$ does not coincide with the universal cover $\tilde{\mathcal{G}}_c(X)$ of $\mathcal{G}_c(X)$. However there exists a natural epimorphism

$$\tilde{\mathcal{G}}_c(X) \rightarrow \tilde{\mathcal{G}}_c(X, V) . \quad (1)$$

Throughout this section we assume that λ is a contact form which obeys the co-orientation and whose Reeb vector field generates a circle action e_t , $t \in S^1$. Denote by $\mathcal{G}_e(V, \lambda)$ the group of contactomorphisms of the form $e_t \cdot \phi$ where $\phi \in \mathcal{G}_c(V)$. We emphasize that by *contactomorphism*, we always mean a diffeomorphism preserving the contact structure, and in this last case, the co-orientation. We do not, however, require contactomorphisms to preserve any specific contact form. We denote by $\tilde{\mathcal{G}}_e$ the universal cover of \mathcal{G}_e .

Observe that when V is a closed manifold, $\mathcal{G}_c(V) = \mathcal{G}_e(V)$, in which case we shall denote it simply by $\mathcal{G}(V)$. When V is an open manifold, every $f \in \mathcal{G}_e(V)$ coincides with some e_t outside a sufficiently large compact subset, so we have a fibration $\mathcal{G}_c(V) \rightarrow \mathcal{G}_e(V) \rightarrow S^1$. The exact homotopy sequence yields in this case that

$$0 = \pi_2(S^1) \rightarrow \pi_1(\mathcal{G}_c) \rightarrow \pi_1(\mathcal{G}_e) ,$$

and hence $\pi_1(\mathcal{G}_c) \rightarrow \pi_1(\mathcal{G}_e)$ is a monomorphism. This implies that $\tilde{\mathcal{G}}_c$ can be considered as a subgroup of $\tilde{\mathcal{G}}_e$.

Let $SV = (V \times \mathbb{R}_+, d(s\lambda))$ be the symplectization of V . For a contactomorphism f of V we write \bar{f} for the corresponding \mathbb{R}_+ -equivariant symplectomorphism of SV . A time-dependent function

$F_t : SV \rightarrow \mathbb{R}$ which is \mathbb{R}_+ -equivariant, i.e. such that $F_t(sx) = sF_t(x)$ for all $s \in \mathbb{R}_+, x \in SV$, is called a *contact Hamiltonian*. The Hamiltonian flow it defines is also \mathbb{R}_+ -equivariant and so produces a contact isotopy of (V, λ) . Moreover, every contact isotopy f_t is given uniquely by such an F_t .

We consider the stabilization $\text{Stab}(SV) = (SV \times T^*S^1, d[s\lambda + rdt])$, and for a subset $X \subset SV$ denote $\text{Stab}(X) = X \times S^1 \subset \text{Stab}(SV)$.

Definition 2.1. We say that a pair (A, B) of closed subsets of SV , with B compact, has *stable intersection property* if $\text{Stab}(A)$ cannot be displaced from $\text{Stab}(B)$ by an \mathbb{R}_+ -equivariant Hamiltonian diffeomorphism of $\text{Stab}(SV)$.

Any $f \in \tilde{\mathcal{G}}_e$ is a homotopy class of path connecting the identity to a fixed contactomorphism. We will write $\{f_t\}_{t \in [a, b]}$ for a specified path in the class f and simply $\{f_t\}$ when $[a, b] = [0, 1]$. We remind that the contact Hamiltonian depends on this choice of path within the class f and is not uniquely defined by f . We will write $H(f_t)$ to denote the Hamiltonian for the path $\{f_t\}$. Recall that the cocycle formula yields

$$H(f_t g_t) = H(f_t) + H(g_t) \circ (\bar{f}_t)^{-1}. \quad (2)$$

The following binary relation introduced and studied in [EP00] plays a crucial role in our story: We write $f \succeq \mathbb{1}$, $f \in \tilde{\mathcal{G}}_e$ if f “can be given by a non-negative contact Hamiltonian”, i.e. there is some path $\{f_t\}$ in the class of f having a non-negative contact Hamiltonian. Observe that in this case $\frac{d}{dt}f_t(x) \in T_{f_t x}$ belongs to or lies on the positive side of the contact hyperplane $\lambda_{f_t x}$. We remark that having a non-negative Hamiltonian is a coordinate-free condition and so $f \succeq \mathbb{1}$ is invariant under conjugation of f in $\tilde{\mathcal{G}}_e$. We write $f \succeq g$ if $f g^{-1} \succeq \mathbb{1}$ (here $f g^{-1}$ corresponds to the class of paths $f_t g_t^{-1}$). By conjugating with g^{-1} , we have the equivalent definition: $f \succeq g$ if $g^{-1} f \succeq \mathbb{1}$. Observe that the relation \succeq is reflexive.

We denote by e the lift of $\{e_t\}_{t \in [0, 1]}$ to $\tilde{\mathcal{G}}_e$, and by e^c , $c \in \mathbb{R}$ the element of $\tilde{\mathcal{G}}_e$ represented by the path $\{e_t\}_{t \in [0, c]}$.

Lemma 2.2. 1. For $f, g \in \tilde{\mathcal{G}}_e$, $f \succeq g$ if and only if f and g can be given by Hamiltonians F and G such that $F \geq G$ (moreover one can fix either the path g_t or f_t); hence \succeq is transitive;
2. $f \succeq g$ and $a \succeq b$ implies that $fa \succeq gb$, i.e. \succeq is bi-invariant;

3. Any element $\phi \in \tilde{\mathcal{G}}_e$ generated by a positive contact Hamiltonian bounded away from zero on the hypersurface $\{s = 1\}$ is dominant, i.e. $\forall f \in \tilde{\mathcal{G}}_e, \exists p \in \mathbb{N}$ s.t. $\phi^p \succeq f$. In particular any e^c , $c > 0$ is dominant.

Proof. To prove the *if* direction of the first property, fix paths $\{f_t\}$ and $\{g_t\}$ for f and g such that $H(f_t) \geq H(g_t)$. By (2)

$$\begin{aligned} H(g_t^{-1}f_t)(x, t) &= H(g_t^{-1})(x, t) + H(f_t)(g_tx, t) \\ &= -H(g_t)(g_tx, t) + H(f_t)(g_tx, t). \end{aligned}$$

So $\{g_t^{-1}f_t\}$ is a path in the class $g^{-1}f$ having non-negative Hamiltonian. Now, to prove the *only if* direction, let $\{h_t\}$ be a path in the class $g^{-1}f$ having non-negative Hamiltonian $H(h_t)$. Let $\{g_t\}$ be an arbitrary path for g and set $f_t = g_th_t$ (or if we wish to choose f_t then set g_t accordingly). Then $h_t = g_t^{-1}f_t$ and so the earlier computation shows that $H(f_t) \geq H(g_t)$.

The second property is also proved using the cocycle formula (2). The third property is straightforward (note that e^c has contact Hamiltonian cs).

□

The next result is a direct analogue of [EP00], Theorem 2.3.A. The essential idea is that anti-symmetry of \succeq can only fail when there exists a non-negative, non-constant contractible loop in $\tilde{\mathcal{G}}_e$ and this is prevented by the stable intersection property. Reflexivity, transitivity and bi-invariance of \succeq are given by the previous Lemma.

Theorem 2.3. *Suppose that SV contains a pair with stable intersection property. Then \succeq is a bi-invariant partial order on $\tilde{\mathcal{G}}_e$.*

Throughout this section we assume that V is orderable; that is the relation \succeq is a partial order on $\tilde{\mathcal{G}}_e(V)$.

Observe that e lies in the center of $\tilde{\mathcal{G}}_e$. For an element $f \in \tilde{\mathcal{G}}_e$ consider the following invariants:

$$\nu_+(f) := \min\{n \in \mathbb{Z} : e^n \succeq f\}$$

and

$$\nu_-(f) := \max\{n \in \mathbb{Z} : e^n \preceq f\}.$$

Note that $\nu_- \leq \nu_+$ by transitivity of \succeq .

It is readily checked that ν_+ and ν_- are conjugation-invariant (since e is in the center) and $\nu_-(f) = -\nu_+(f^{-1})$, using the bi-invariance of \succeq . We also observe that ν_+ and ν_- are respectively sub- and super-additive:

$$\begin{aligned}\nu_+(fg) &\leq \nu_+(f) + \nu_+(g) \\ \nu_-(fg) &\geq \nu_-(f) + \nu_-(g).\end{aligned}$$

We write

$$\nu(f) := \max(|\nu_+(f)|, |\nu_-(f)|).$$

This gives the following.

Theorem 2.4. *ν is a conjugation-invariant norm on $\tilde{\mathcal{G}}_e$.*

Proof. Clearly $\nu \geq 0$. Observe moreover that $\nu(f) = 0$ if and only if $\mathbb{1} \succeq f \succeq \mathbb{1}$ and hence $f = \mathbb{1}$. Since both ν_+ and ν_- are conjugation-invariant, ν is as well. It remains to prove the triangle inequality. Let $f, g \in \tilde{\mathcal{G}}_e$. We consider 2 cases.

Case 1: $|\nu_-(fg)| < |\nu_+(fg)|$ so $\nu(fg) = |\nu_+(fg)|$. We claim that $\nu_+(fg) \geq 0$ so that $\nu(fg) = \nu_+(fg)$ and the result follows from the sub-additivity of ν_+ . To prove the claim, consider the two cases $\nu_-(fg) \geq 0$ or $\nu_-(fg) < 0$. In the first case, $\nu_- \leq \nu_+$ immediately implies $\nu_+(fg) \geq 0$, in the second case, this is also true, because $|\nu_-(fg)| < |\nu_+(fg)|$ prevents $\nu_+(fg)$ from being negative without lying below $\nu_-(fg)$.

Case 2: $|\nu_-(fg)| \geq |\nu_+(fg)|$ so either $\nu_-(fg) = \nu_+(fg) > 0$ in which case $\nu(fg) = \nu_+(fg)$ and we handle as above or $\nu_-(fg) \leq 0$, in which case $\nu(fg) = |\nu_-(fg)| = -\nu_-(fg)$, and the result follows from the super-additivity of ν_- . \square

Theorem 2.5. *Suppose that (A, B) has stable intersection property and in addition B is an invariant subset for the flow \bar{e}_t . Assume that for some $f \in \tilde{\mathcal{G}}_e$, the element \bar{f} is generated by a contact Hamiltonian F_t which satisfies $F_t > cs$ on B , $c > 0$ for all $t \in S^1$. Then $\nu_+(f) \geq [c]$, the integer-part of c , and so $\nu(f) \geq [c]$.*

Consider now the restriction of the norm ν to the subgroup $\tilde{\mathcal{G}}_c(V) \subset \tilde{\mathcal{G}}_e(V)$. The contact Hamiltonian F_t in Theorem 2.5 can be taken in

such a way that it generates a compactly supported contact isotopy. Since f^k with $k \in \mathbb{N}$ is generated by the contact Hamiltonian kF_{kt} , we see that $\nu(f^k) \geq [c] \cdot k$. We conclude that in the setting of the theorem, the group $\mathcal{G}_c(V)$ equipped with the norm ν is stably unbounded. Moreover, choosing F to be time-independent so that $F \leq Cs$ on SV with $C > c$ we get that the Hamiltonian flow $t \mapsto f_t$ of F defines a quasi-isometric monomorphism $\mathbb{R} \rightarrow \widetilde{\mathcal{G}}_c(V)$, as stated in the abstract.

Before proving the theorem, let us recall the following construction (see [EP00]). Let $\varphi = \{\phi_t\}$, $t \in S^1$, $\phi_0 = \phi_1 = \mathbb{1}$ be a loop of contactomorphisms from \mathcal{G}_e of a contact manifold V generated by a \mathbb{R}_+ -equivariant Hamiltonian Φ_t on SV . Define the suspension map

$$\Sigma_\varphi : SV \times T^*S^1 \rightarrow SV \times T^*S^1$$

of φ by

$$(z, r, t) \mapsto (\overline{\phi}_t z, r - \Phi_t(\overline{\phi}_t z), t) .$$

The map Σ_φ is an \mathbb{R}_+ -equivariant symplectomorphism of $SV \times T^*S^1$. Furthermore, if φ is contractible, and $\varphi^{(s)}$ is the homotopy of $\varphi = \varphi^{(0)}$ to the constant loop $\varphi^{(1)} = \mathbb{1}$, the family of the suspension maps $\Sigma_{\varphi^{(s)}}$ is a Hamiltonian isotopy of $SV \times T^*S^1$.

Lemma 2.6. *Let $A, B \subset SV$ be a pair of subsets with stable intersection property. Then for every contractible loop $\varphi = \{\phi_t\}$ its contact Hamiltonian Φ vanishes for some $t_0 \in S^1$ and $y \in B$: $\Phi_{t_0}(y) = 0$.*

Proof. The stable intersection property implies that the sets $\Sigma_\varphi(A \times S^1)$ and $B \times S^1$ intersect. Thus there exists $z \in A$ and t_0 so that $\overline{\phi}_{t_0} z \in B$ and $\Phi_{t_0}(\overline{\phi}_{t_0} z) = 0$. Setting $y = \overline{\phi}_{t_0} z$, we get the lemma. \square

Proof of Theorem 2.5: Assume on the contrary that $\nu_+(f) < c$ for some integer c . Then $f \preceq e^c$ (recall e^c denotes the class of the path $\{e_{ct}\}$). This means that

$$H(f_t) \leq H(e_{ct}\phi_t) \tag{3}$$

for some contractible loop $\varphi = \{\phi_t\}$ on \mathcal{G}_e . By the cocycle formula, this yields $F_t \leq cs + \Phi_t \circ e_{-ct}$. Applying Lemma 2.6, we see that $\Phi_{t_0}(y) = 0$ for some $t_0 \in S^1$ and $y \in B$. Since $x := e_{ct}y \in B$, we get that $F_t(x) \leq cs(x)$, and we get a contradiction with the assumption $F_t|_B > cs$. \square

If we consider only compactly supported contactomorphisms, i.e. restrict the above norm to $\widetilde{\mathcal{G}}_c$, it descends to \mathcal{G}_c as follows. Given $f' \in \mathcal{G}_c$, define

$$\nu_*(f') := \inf \nu(f) , \quad (4)$$

where the infimum is taken over all lifts f of f' to $\widetilde{\mathcal{G}}_c \subset \widetilde{\mathcal{G}}_e$. Observe that ν_* is non-degenerate: indeed, since $\nu(f)$ is integer, the infimum is necessarily attained on some lift f . But $\nu(f) = 0$ yields $f = \mathbb{1}$ and hence $f' = \mathbb{1}$.

Denote by Π the image of the fundamental group $\pi_1(\mathcal{G}_e, \mathbb{1})$ in $\pi_1(\mathcal{G}_e, \mathbb{1})$ under the natural inclusion morphism. Each loop in \mathcal{G}_e representing an element of Π can be written as a product of a contractible loop in \mathcal{G}_e and a loop in \mathcal{G}_c (note that the order of factors is not important since \mathcal{G}_c is a normal subgroup of \mathcal{G}_e).

Given a pair of subsets $A, B \subset SV$, where B is compact, we say that it has *strong stable intersection property* if for every loop φ representing an element of Π its suspension Σ_φ satisfies

$$\Sigma_\varphi(A \times S^1) \cap (B \times S^1) \neq \emptyset . \quad (5)$$

Theorem 2.7. *Suppose that (A, B) has strong stable intersection property and in addition B is an invariant subset for the flow \bar{e}_t . Assume that for some $f \in \widetilde{\mathcal{G}}_c$, the element \bar{f} is generated by a contact Hamiltonian F_t which satisfies $F_t > cs$ on B , $c > 0$ for all $t \in S^1$. Let $f' \in \mathcal{G}_c$ be the time-one map of the Hamiltonian flow f_t . Then $\nu_*(f') \geq c$.*

The proof repeats verbatim that of Theorem 2.5 with the following modifications: Lemma 2.6 holds for loops φ representing an element from Π , and inequality (3) holds true with the loop $\{\phi_t\}$ representing an element of Π .

Example 2.8. Assume that $V = T^*X \times S^1$, where X is a closed manifold, $\lambda = d\tau - pdq$, $e_t(p, q, \tau) = (p, q, \tau + t)$. For a domain $\mathcal{U} \subset T^*X$ put $\widehat{\mathcal{U}} := \mathcal{U} \times S^1$. Observe that

$$\Psi : SV \rightarrow T^*X \times T^*S^1, (p, q, \tau, s) \mapsto (-sp, q, \tau, s)$$

is an \mathbb{R}_+ -equivariant symplectic embedding with respect to the standard \mathbb{R}_+ -actions. The Lagrangian submanifold

$$A = B = X \times \{s = 1\} \subset SV$$

is stably non-displaceable by the standard Floer theory. Thus ν is a metric on $\widetilde{\mathcal{G}}_e$ in this case. Furthermore, Theorem 2.7 is applicable to this situation since $\Sigma_\varphi(B)$ has the same Liouville class as B and hence these Lagrangian submanifolds intersect by a theorem of Gromov [G85, 2.3.B''₄]. Therefore the pair (B, B) has strong stable intersection property. We conclude that the norm ν_* defined by (4) is unbounded on the subgroup $\mathcal{G}_c(\widehat{\mathcal{U}}, V)$ consisting of all contactomorphisms generated by contact isotopies with support in $\widehat{\mathcal{U}}$, where $\mathcal{U} \subset T^*X$ is any tube containing the zero section. In fact, the group $\mathcal{G}_c(V)$ is stably unbounded with respect to ν_* . Let us mention also that the norm ν_* is greater than or equal to the norm defined by Zapolsky in [Z12] (this readily follows from [Z12]).

Example 2.9. Let as above $V = T^*X \times S^1$. Under extra hypotheses, one can make an even stronger statement than just unboundedness of our norm on $\widetilde{\mathcal{G}}_c(\widehat{\mathcal{U}}, V)$ for tube \mathcal{U} (cf. Example 2.8). We now describe a setting in which $\widetilde{\mathcal{G}}_c(\widehat{\mathcal{U}}, V)$ with norm ν admits a quasi-isometric monomorphism of \mathbb{R}^N for any N . Let $L \subset T^*X$ be a closed Lagrangian submanifold such that

- (a) $HF(L, L) \neq 0$ (Floer homology with coefficients in a field, say \mathbb{Z}_2)
- (b) $(a \cdot L) \cap L = \emptyset$, $\forall a > 0, a \neq 1$, where $a \cdot (p, q) = (ap, q)$.

We claim that for any bounded domain $\mathcal{U} \subset T^*X$ containing the zero section and any $N \in \mathbb{N}$, $\widetilde{\mathcal{G}}_c(\widehat{\mathcal{U}}, V)$ admits a quasi-isometric monomorphism of \mathbb{R}^N . Some examples of X and L as above:

1. X is a closed manifold admitting a closed 1-form α without zeroes, and L is the graph of α .
2. $X = S^2$ and L is the Lagrangian torus studied by Albers and Frauenfelder in [AF08] with $HF(L, L; \mathbb{Z}_2) \neq 0$.

Proof. Fix $N \in \mathbb{N}$ and fix a bounded tube \mathcal{U} around the zero section. Choose distinct real numbers $a_1, \dots, a_N, a_j \neq 1$ such that $L_j := a_j L \subset \mathcal{U}$. Thus L_j , $j = 1, \dots, N$ are pairwise disjoint. In $S(T^*X \times S^1) \subset T^*X \times T^*S^1$ put $\hat{L}_j := L_j \times \{s = 1\}$. Let W_j be tubular neighborhoods of \hat{L}_j respectively such that $\overline{W}_j \cap \overline{W}_i = \emptyset$ when $i \neq j$.

Take contact Hamiltonians H_j , with $\text{supp } H_j \subset W_j$, $H_j = 1$ on \hat{L}_j and $0 \leq H_j \leq s$. Let h_j^t be the corresponding Hamiltonian flow. Consider the map $\Psi : \mathbb{R}^N \rightarrow \widetilde{\mathcal{G}}_c(\mathcal{U}, V)$ given by $(t_1, \dots, t_N) \mapsto h_1^{t_1} \dots h_N^{t_N}$.

This is by construction a homomorphism, which is injective since W_j 's are pair-wise disjoint. On the one hand, $h_1^{t_1} \dots h_N^{t_N}$ is generated by $H = \sum_{j=1}^N t_j H_j$ so

$$\nu(h_1^{t_1} \dots h_N^{t_N}) \leq \max_j (|t_j| + 1).$$

On the other hand, $H|_{\hat{L}_j} = t_j$ so by Theorem 2.5

$$\nu(h_1^{t_1} \dots h_N^{t_N}) \geq \max_j (|t_j| - 1).$$

Thus

$$||t||_\infty - 1 \leq \nu(h_1^{t_1} \dots h_N^{t_N}) \leq ||t||_\infty + 1 ,$$

where $||t||_\infty := \max_j |t_j|$. We conclude that

$$\Psi : (\mathbb{R}^N, ||\cdot||_\infty) \rightarrow (\tilde{\mathcal{G}}_c(\mathcal{U}, V), \nu)$$

is a quasi-isometric monomorphism. \square

Example 2.10. Let (M, ω) be a closed symplectic manifold with $[\omega] \in H^2(M, \mathbb{Z})$. Let $\pi : V \rightarrow M$ be a prequantization of M . This means that π is a principal S^1 -bundle equipped with an S^1 -invariant contact form λ such that $d\lambda = \pi^*\omega$. The Reeb flow e_t of λ is just the natural S^1 -action on (M, ω) . We shall focus on the group $\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_c = \tilde{\mathcal{G}}_e$ (the last two groups coincide since M is closed).

First, assume that M contains a closed Lagrangian submanifold L with the following properties:

- (i) The connection on V defined by λ has trivial holonomy when restricted to L (the Bohr-Sommerfeld condition);
- (ii) The relative homotopy group $\pi_2(M, L)$ vanishes.

In view of (i), L admits a Legendrian lift $\hat{L} \subset V$. Put

$$A := \hat{L} \times \mathbb{R}_+ \subset SV, \quad B = \pi^{-1}(L) .$$

By Example 2.4.B of [EP00], the pair (A, B) has stable Lagrangian intersection property. Furthermore, B is invariant under \bar{e}_t , and hence by Theorems 2.4 and 2.5 the norm ν on $\tilde{\mathcal{G}}(V)$ is well defined and unbounded.

Second, assume that $M = S^2$ of total area 1, and $V = \mathbb{R}P^3$. Observe that SV is symplectomorphic to T^*S^2 with the zero section removed. Let $L \subset T^*S^2$ be the monotone Lagrangian torus considered in [AF08] (cf. Example 2.9 above). One readily checks that L is invariant under \bar{e}_t . Since the pair (L, L) has stable intersection property, we conclude from Theorems 2.4 and 2.5 that the norm ν on $\tilde{\mathcal{G}}(\mathbb{R}P^3)$ is well defined and unbounded. The situation changes drastically when we pass to the double cover S^3 of $\mathbb{R}P^3$: as we shall see in the next section, any conjugation-invariant norm on $\tilde{\mathcal{G}}(S^3)$ is bounded.

3 Obstructions

3.1 Triviality of norms for S^{2n+1}

Let S^{2n+1} be the standard contact sphere of dimension ≥ 3 . In Sections 3.1-3.4 we will be dealing with the group $\mathcal{G} = \mathcal{G}(S^{2n+1})$ and its universal cover $\tilde{\mathcal{G}}$.

Theorem 3.1.

1. *Any conjugation-invariant norm on \mathcal{G} is bounded and discrete.*
2. *Any conjugation-invariant norm on $\tilde{\mathcal{G}}$ is bounded. On $\tilde{\mathcal{G}} \setminus \pi_1(\mathcal{G})$, it is also bounded away from 0.*

Example 3.2. The next example shows that the second part of Theorem 3.1 cannot be improved. It follows from [E92] that $\pi_1(\mathcal{G}(S^3)) = \mathbb{Z}$. Let $\phi \in \pi_1(\mathcal{G}(S^3))$ be a generator, and let $r \in [0, 1]$ be an irrational number. Define a norm ν on $\tilde{\mathcal{G}}(S^3)$ by setting

$$\nu(\phi^n) = |e^{2\pi i n r} - 1|,$$

and $\nu(g) = 1$ for $g \notin \pi_1(\mathcal{G}(S^3))$. One readily checks that ν defines a norm on $\tilde{\mathcal{G}}(S^3)$, which is conjugation-invariant since $\pi_1(\mathcal{G}(S^3))$ is a normal subgroup. Moreover, ν is clearly not discrete.

Since the proofs of the discreteness and boundedness in Theorem 3.1 are independent we present them separately, in Sections 3.3 and 3.4 respectively. First, in Section 3.2 we quote the algebraic results used in the proofs. We conclude the paper with a discussion on a class of sub-domains of contact manifolds for which the restriction of any conjugation-invariant norm is necessarily bounded.

3.2 Algebraic results

Here we recall the necessary algebraic results which are essential to our proof. The first is taken from [BIP08]. To state it we need the following definition, from the same paper.

Definition 3.3. Let G be a group, and let $H \subset G$ be a subgroup. We say that an element $g \in G$ *m-displaces* H if the subgroups

$$H, gHg^{-1}, g^2Hg^{-2}, \dots, g^mHg^{-m}$$

pairwise commute.

The geometric meaning of *m-displacement* in our context is as follows. Let $U \subset S^{2n+1}$ be an open subset. We say that a contactomorphism $\phi \in \mathcal{G}$ *m-displaces* U if the subsets

$$U, \phi(U), \dots, \phi^m(U)$$

are pairwise disjoint. If this holds then ϕ *m-displaces* the subgroup $\mathcal{G}(U)$ of \mathcal{G} . Similarly, if $\tilde{\phi} = \{\phi_t\} \in \tilde{\mathcal{G}}$ is a path such that ϕ_1 *m-displaces* U then $\tilde{\phi}$ *m-displaces* the subgroup $\tilde{\mathcal{G}}(U, S^{2n+1})$ of $\tilde{\mathcal{G}}(S^{2n+1})$.

Returning to the general algebraic setting, given a subgroup $H \subset G$ and an element h in the commutator subgroup $[H, H]$, we denote by $cl_H(h)$ the commutator length of h , which is the minimal number of commutators needed to represent h as a product of commutators. We shall need the following result (see [BIP08], Theorem 2.2). Suppose that ν is a conjugation-invariant norm on a group G . Let $H \subset G$ be a subgroup such that there exists $g \in G$ which *m-displaces* H . Then for any $h \in [H, H]$ with $cl_H(h) = m$ one has

$$\nu(h) \leq 14\nu(g). \tag{6}$$

Finally, we will use the following result of Rybicki [R10]. Let W be any contact manifold of dimension ≥ 3 . Then the groups $\mathcal{G}_c(W)$ and $\tilde{\mathcal{G}}_c(W)$ are perfect, i.e. equal to their commutator subgroups. In particular if $X \subset V$ is an open subset, $\mathcal{G}_c(X)$ and $\tilde{\mathcal{G}}_c(X, V)$ are perfect. The latter group is perfect since it is an epimorphic image of a perfect group (see (1) above).

3.3 Discreteness

Here we prove the discreteness in Theorem 3.1. In what follows by an embedded open ball we mean the interior of an embedded closed

ball. We shall use the fact that given any two embedded balls $B, D \subset S^{2n+1}$ one can find a contactomorphism $\eta \in \mathcal{G}$ such that $\eta(D) \subset B$, see Example 3.7 below.

Lemma 3.4. *Let ν be a conjugation-invariant norm on \mathcal{G} or $\tilde{\mathcal{G}} \setminus \pi_1(\mathcal{G})$. Then ν is bounded away from 0.*

Proof. We give the proof for $\tilde{\mathcal{G}}$. Assume the result does not hold: For any $\varepsilon > 0$ we can find $\theta = \{\theta_t\} \in \tilde{\mathcal{G}} \setminus \pi_1(\mathcal{G})$ with $\nu(\theta) < \varepsilon$. Since $\theta_1 \neq \mathbb{1}$, θ_1 moves some point, so there must exist an open ball $B \subset S^{2n+1}$ such that $\theta_1(B) \cap B = \emptyset$.

Fix now a ball D and a pair of elements $\phi, \psi \in \tilde{\mathcal{G}}(D, S^{2n+1})$ with $[\phi, \psi] \neq \mathbb{1}$. Let $\eta = \{\eta_t\} \in \tilde{\mathcal{G}}$ such that $\eta_1(D) \subset B$. Then $\eta^{-1}\theta\eta$ displaces $\tilde{\mathcal{G}}(D, S^{2n+1})$, and hence by (6)

$$\nu([\phi, \psi]) \leq 14\nu(\eta^{-1}\theta\eta) = 14\nu(\theta) < 14\varepsilon.$$

Since ε was arbitrary, this implies $\nu([\phi, \psi]) = 0$, contradicting the non-degeneracy of ν . \square

3.4 Boundedness

Here we prove the upper bound in Theorem 3.1.

Lemma 3.5. *Let ν be a conjugation-invariant norm on \mathcal{G} or $\tilde{\mathcal{G}}$. Then ν is bounded.*

We prove the case of $\tilde{\mathcal{G}}$, since the proof for \mathcal{G} is similar. Therefore we assume from now on that ν is a conjugation-invariant norm on $\tilde{\mathcal{G}}$.

Definition 3.6. An open contact manifold (V, ξ) is called *contact portable* if there exists a compact set $V_0 \subset V$ and a contact isotopy $\{P_t\}$ of V , $t \geq 0$, $P_0 = \mathbb{1}$ such that following hold:

- The set V_0 is an *attractor* of $\{P_t\}$, i.e. for every compact set $K \subset V$ there exists some $t > 0$ such that $P_t(K) \subset V_0$.
- There exists a contactomorphism θ of V displacing V_0 .

Let us mention that θ is not assumed to be compactly supported. Definition 3.6 is a contact version of the notion of portable manifold defined in [BIP08].

Example 3.7. An example of a contact portable manifold is \mathbb{R}^{2n+1} , equipped with the standard contact structure given by the kernel of the 1-form $\alpha = dz - ydx$. Here we use the coordinates $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. The contact isotopy is given by

$$P_t: (x, y, z) \mapsto (e^{-t}x, e^{-t}y, e^{-2t}z).$$

The attractor V_0 can be taken to be the closed ball $\{|x|^2 + |y|^2 + z^2 \leq 1\}$ and the contactomorphism θ can be given by

$$\theta(x, y, z) = (x, y, z + 3),$$

for example. In particular, this example shows that every compact subset of \mathbb{R}^{2n+1} can be contactly isotoped into an arbitrary small neighborhood of the origin. Since $S^{2n+1} \setminus \text{point}$ is contactomorphic to \mathbb{R}^{2n+1} , this yields the claim made in the beginning of Section 3.3: every ball in the sphere can be contactly isotoped into any given ball.

The proof of the following proposition is identical to the proof of Theorem 1.17 in [BIP08].

Proposition 3.8. *Let (V, ξ) be a contact portable manifold. Then any conjugation-invariant norm on $\mathcal{G}_c(V)$ or $\tilde{\mathcal{G}}_c(V)$ is bounded.*

Proof. We only give a sketch of the proof, since it is taken verbatim from [BIP08]. The contactomorphism θ displaces a neighborhood U of V_0 . Then $W = \theta(U)$ is an attractor for the isotopy $\{g_t = \theta \circ P_t \circ \theta^{-1}\}$. Therefore, for some $T > 0$, $g_T(U \cup W) \subset W$. Truncating the contact Hamiltonian generating $\{g_t\}$ and re-parametrizing gives a contact isotopy $\psi = \{\psi_t\} \in \tilde{\mathcal{G}}_c(V)$ such that $\psi_1(U \cup W) \subset W$. It is easily verified that ψ_1 m -displaces U for all $m \geq 0$. Indeed $\psi_1(U)$ lies strictly between W and $\psi_1(W) \subset W$ (i.e. $\psi_1(U) \subset W \setminus \psi_1(W)$) and so $\forall k \in \mathbb{N}$, $\psi_1^k(U) \subset \psi_1^{k-1}(W) \setminus \psi_1^k(W)$, which implies the $\psi_1^k(U)$ are pairwise disjoint. Now, by Rybicki's theorem the subgroup $H := \tilde{\mathcal{G}}(U, V)$ is perfect, and hence any $h \in H$ has $cl_H(h) = m$ for some $m < \infty$. Therefore for any $h \in H$,

$$\nu(h) \leq 14\nu(\psi).$$

Moreover, given any $h = \{h_t\} \in \tilde{\mathcal{G}}_c(V)$ with $\cup_t \text{supp } h_t \subset K$ for a compact K , we can find T such that $P_T(K) \subset U$. Thus we can produce $\eta \in \tilde{\mathcal{G}}_c(V)$ with $\eta(K) \subset U$. Then the isotopy $k = \{k_t = \eta \circ h_t \circ \eta^{-1}\}$ lies in $\tilde{\mathcal{G}}(U, V)$ and thus

$$\nu(h) = \nu(k) \leq 14\nu(\psi).$$

□

In the next proposition put $V = S^{2n+1}$ and $V_z := V \setminus \{z\}$ for a point $z \in V$.

Proposition 3.9. *Any contactomorphism $f \in \mathcal{G}(V)$ can be written as $f = gh$, where $g \in \mathcal{G}(V_z)$ and $h \in \mathcal{G}(V_w)$ for some $z, w \in V$. Similarly, any element $\tilde{f} \in \tilde{\mathcal{G}}(V)$ can be written as $\tilde{f} = \tilde{g}\tilde{h}$, where $\tilde{g} \in \tilde{\mathcal{G}}(V_z, V)$ and $\tilde{h} \in \tilde{\mathcal{G}}(V_w, V)$ for some points $z, w \in V$.*

Proof. Let f_t be a contact isotopy generating f . Take a sufficiently small ball $B \subset V$ such that $X := \cup_t f_t(B) \neq S^{2n+1}$. Fix any point $z \notin X$. Let $\{g_t\}$ be a contact isotopy supported in V_z with $g_t|_B = f_t|_B$ for all $t \in [0, 1]$. Set $h_t = g_t^{-1}f_t$. Observe that $h_t \in \mathcal{G}(V_w)$ for any point $w \in B$. Then $\tilde{f} = \tilde{g}\tilde{h}$ and $f_1 = g_1h_1$ are the desired decompositions. \square

Proof of Lemma 3.5: We use the notation of Proposition 3.9. Observe that $S^{2n+1} \setminus \{point\}$ is contactomorphic to \mathbb{R}^{2n+1} , and hence the norm ν is bounded on $\mathcal{G}_c(V_z)$ and $\mathcal{G}_c(V_w)$ by Proposition 3.8. Therefore, by Proposition 3.9, it is bounded on $\mathcal{G}(S^{2n+1})$.

Further, the groups $\tilde{\mathcal{G}}_c(V_z, V)$ and $\tilde{\mathcal{G}}_c(V_w, V)$ are epimorphic images of $\tilde{\mathcal{G}}_c(V_z)$ and $\tilde{\mathcal{G}}_c(V_w)$, see (1). Since every conjugation-invariant norm is bounded on the latter groups (use Proposition 3.8), the same holds true for the former groups by Lemma 1.10 in [BIP08]. Applying again Proposition 3.9, we conclude that ν is bounded on $\mathcal{G}(S^{2n+1})$. \square

This completes the proof of Theorem 3.1.

3.5 Boundedness on sub-domains

Definition 3.10. Suppose (V, ξ) is a contact manifold and $V' \subset V$ an open subset. Then we say that V' is a *portable sub-domain* of V if there exists a compact set $V_0 \subset V'$, and a contact isotopy $\{P_t\}_{t \in \mathbb{R}}$ of V such that following hold:

- For every compact set $K \subset V'$ there exists some $t > 0$ such that $P_t(K) \subset V_0$.
- There exists a contactomorphism θ supported in V' displacing V_0 .

This notion is strictly weaker than V' being a contact portable manifold in its own right, since we allow more “squeezing room” (the isotopy P_t may have support outside V'). By the same argument in the proof of Proposition 3.8, we have the following result:

Proposition 3.11. *Let (V, ξ) be a contact manifold and $V' \subset V$ be a portable sub-domain of V . Then any conjugation-invariant norm on $\mathcal{G}_c(V)$ (resp. $\tilde{\mathcal{G}}_c(V)$) is necessarily bounded on $\mathcal{G}_c(V')$ (resp. on $\tilde{\mathcal{G}}_c(V', V)$).*

Example 3.12. Consider the contact manifold $V = \mathbb{R}^{2n} \times S^1$ equipped with the contact structure $\xi = \text{Ker}(dt - \alpha)$, where $\alpha = \frac{1}{2}(pdq - qdp)$, and write simply \mathcal{G}_c for $\mathcal{G}_c(V)$. In what follows we assume that $n \geq 2$. Put $\mathcal{U}(r) := B^{2n}(r) \times S^1$, where $B^{2n}(r)$ stands for the ball $\{\pi(|p|^2 + |q|^2) < r\}$. In [S10] Sandon defined a conjugation invariant norm on $\mathcal{G}_c(V)$ which is bounded on all subgroups $\mathcal{G}_c(\mathcal{U}(r))$. We claim that every conjugation-invariant norm on $\mathcal{G}_c(V)$ is necessarily bounded on $\mathcal{G}_c(\mathcal{U}(r))$ if $r < 1$.

To prove the claim, take any Hamiltonian symplectomorphism θ supported in $B^{2n}(r) \subset \mathbb{R}^{2n}$ which displaces the origin. Let $r' < r$ be sufficiently small so that $B^{2n}(r')$ is also displaced and let $\hat{\theta} \in \mathcal{G}_c(\mathcal{U}(r))$ be the lift of θ to a contactomorphism of V supported in $\mathcal{U}(r)$. We have $\hat{\theta}(\mathcal{U}(r')) \cap \mathcal{U}(r') = \emptyset$. Further, by the Squeezing Theorem [EKP06, Theorem 1.3] there exists $P \in \mathcal{G}_c(V)$ such that $P(\mathcal{U}(r)) \subset \mathcal{U}(r')$. Therefore $\mathcal{U}(r)$ is a contact portable sub-domain of V , and the claim follows from Proposition 3.11.

In fact, the above argument can be applied more generally. Recall that a symplectic manifold $(M^{2n}, \omega = d\alpha)$ is called *Liouville* if it admits a vector field η and a compact $2n$ -dimensional submanifold $\overline{\mathcal{U}}$ with connected boundary $Q = \partial\overline{\mathcal{U}}$ with the following properties:

- $i_\eta \omega = \alpha$. This yields that the flow η^t of η is conformally symplectic;
- η is transversal to Q .

One can show that $(Q, \text{Ker}(\alpha))$ is a contact manifold which does not depend on the specific choice of Q . It is called *the ideal contact boundary* of M . The set $C := \bigcap_{t>0} \eta^{-t}(\overline{\mathcal{U}})$ is called *the core* of M .

Consider now the contact manifold $V = M \times S^1$ equipped with the contact form $\lambda = dt - \alpha$. Put $\mathcal{U}(r) = \eta^{-r}(\mathcal{U}) \times S^1$, where \mathcal{U} is the interior of $\overline{\mathcal{U}}$.

Proposition 3.13. *Suppose that the ideal contact boundary of $(M, d\alpha)$ is non-orderable and the core C is displaceable by a Hamiltonian diffeomorphism in its arbitrary small neighborhood. Then there exists*

$r_0 > 0$ so that any conjugation-invariant norm on $\mathcal{G}_c(V)$ (resp. $\tilde{\mathcal{G}}_c(V)$) is necessarily bounded on $\mathcal{G}_c(\mathcal{U}(r))$ (resp. on $\tilde{\mathcal{G}}_c(\mathcal{U}(r), V)$) for all positive $r < r_0$.

Proof. By [EKP06, Theorem 1.19] there is some $r_0 > 0$ such that (by iterating the Theorem enough times), one can obtain an isotopy which squeezes $\mathcal{U}(r_0)$ arbitrarily close to $C \times S^1$, in fact within¹ some larger $\mathcal{U}(r') \subset V$. Since the core C is Hamiltonian displaceable in its arbitrarily small neighborhood, the set $C \times S^1$ is displaceable in its arbitrary small neighborhood by a contact isotopy of V . It follows that for $0 < r < r_0$ the set $\mathcal{U}(r)$ is a portable sub-domain of V and hence by Proposition 3.11 any conjugation-invariant norm on $\mathcal{G}_c(V)$ (resp. $\tilde{\mathcal{G}}_c(V)$) is bounded on $\mathcal{G}_c(\mathcal{U}(r))$ (resp. on $\tilde{\mathcal{G}}_c(\mathcal{U}(r), V)$). \square

An important class of Liouville manifolds is formed by complete Stein manifolds, that is by Kähler manifolds $(M, J, d\alpha)$ admitting a proper bounded from below Morse function F with $\alpha = JdF$ and $\eta = -\text{grad}F$, where the gradient is taken with respect to the metric $d\alpha(\cdot, J\cdot)$. (By a result of Eliashberg [E90] these manifolds admit an alternative description as Weinstein manifolds provided $\dim M \geq 3$). For a generic F , the core of M is an isotropic CW -complex of dimension $n-k$ with $k \in [0; n]$ (see [EG91, BC02]). We say that M is k -subcritical if $k \geq 1$ and *critical* if $k = 0$. The assumptions of Proposition 3.13 hold true, for instance, when the Liouville manifold M is k -critical with $k \geq 2$: indeed, the ideal contact boundary is non-orderable by Theorem 1.16 of [EKP06], while the core C in this case is Hamiltonian displaceable in its arbitrary small neighborhood (cf. [BC02, Section 3]).

The case of minimal possible subcriticality $k = 1$ is currently out of reach. It is unknown whether the ideal contact boundary of such an M is orderable or not. Furthermore it is unlikely that the technique of Section 2 based on stable intersection property might be applicable to subcritical manifolds with $\dim M \geq 2$: Indeed, by [BC02a, Theorem 6.1.1], there are no “hard” symplectic obstructions to Hamiltonian displacement of a compact subset of $SV = M \times \mathbb{R}_+ \times S^1$ from a given closed subset, so existence of pairs with stable intersection property is quite problematic. It would be interesting to explore this point further.

¹See the argument in Remark 1.23 of that paper and the proof of Theorem 1.3 on the same page.

We conclude the paper with a couple of open questions.

Question 3.14. Is it true that the core of a critical Stein manifold is necessarily stably non-displaceable?

For instance, this is true for cotangent bundles of closed manifolds, where the core can be taken as the zero section. By Theorems 2.4 and 2.5 the affirmative answer would yield that the conjugation invariant norm ν on $\widetilde{\mathcal{G}}_c(V)$ in this case is well defined and unbounded when restricted to $\widetilde{\mathcal{G}}_c(\mathcal{U}(r))$ for all $r > 0$. Here the 1-periodic Reeb flow on V is associated to the form λ and is simply given by the rotation along the S^1 -factor.

In fact, for general Liouville manifolds it is still unclear whether we even can start developing the theory of Section 2:

Question 3.15. Is it true that (V, λ) is orderable for a every Liouville manifold M ?

This is true, for instance, for some critical M such as cotangent bundles, as well as for some n -subcritical M such as linear spaces due to the thesis of Sandon [S11]. P. Albers pointed out that the methods of [AM12] should yield an affirmative answer to this question.

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